Math 150, *Lecture Notes- Bonds* Name____________

Section 1.3 *Evaluating Limits Analytically*

Properties of Limits:

THEOREM 1.1 Some Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$ **2.** $\lim_{x \to c} x = c$ **3.** $\lim_{x \to c} x^n = c^n$

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$

2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$

5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Use ***basic limits and properties of limits*** to find the following limits.

Ex.1 $\lim_{x \to 1}\left(12x^3 - 6x + 5\right) = \lim_{x \to 1}(12x^3) - \lim_{x \to 1}(6x) + \lim_{x \to 1}(5)$

$= 12 \cdot \lim_{x \to 1}(x^3) - 6 \cdot \lim_{x \to 1}(x) + 5$

$= 12 \cdot [\lim_{x \to 1}(x)]^3 - 6 \cdot [1] + 5$

$= 12 \cdot [1]^3 - 6 + 5$

$= 12 \cdot 1 - 1$

$= 12 - 1$

$= 11$

$\lim_{x \to 1}(12x^3 - 6x + 5)$

$= 12(1)^3 - 6(1) + 5$

$= 12 - 6 + 5$

$= 11$

Ex.2 $\lim\limits_{x\to 1}\dfrac{3x+5}{x+1} = \dfrac{\lim\limits_{x\to 1}(3x+5)}{\lim\limits_{x\to 1}(x+1)}$

$= \dfrac{\lim\limits_{x\to 1}(3x) + \lim\limits_{x\to 1} 5}{\lim\limits_{x\to 1}(x) + \lim\limits_{x\to 1} 1}$

$= \dfrac{3\cdot\lim\limits_{x\to 1}(x) + 5}{1 + 1}$

$= \dfrac{3\cdot 1 + 5}{2}$

$= \dfrac{8}{2}$

$= 4$

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim\limits_{x\to c} g(x) = L$ and $\lim\limits_{x\to L} f(x) = f(L)$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

Ex.3 $\lim_{x\to -3}\sqrt[3]{12x+3} = \sqrt[3]{\lim_{x\to -3}(12x+3)}$

$= \sqrt[3]{\lim_{x\to -3}(12x) + \lim_{x\to -3}(3)}$

$= \sqrt[3]{12\cdot \lim_{x\to -3}(x) + 3}$

$= \sqrt[3]{12(-3)+3}$

$= \sqrt[3]{-36+3}$

$= -\sqrt[3]{33}$

$\lim_{x\to -3}\sqrt[3]{12x+3}$

$= \sqrt[3]{\lim_{x\to -3}(12x+3)}$

$= \sqrt[3]{12(-3)+3}$

$= -\sqrt[3]{33}$

Ex.4 Given $\lim_{x\to c} f(x) = 27$, evaluate the following limits:

(a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)}$

$= \sqrt[3]{27}$

$= 3$

(b) $\lim_{x\to c}\frac{f(x)}{18} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18}$

$= \frac{27}{18}$

$= \frac{3}{2}$

(c) $\lim_{x\to c}[f(x)]^2 = \left(\lim_{x\to c} f(x)\right)^2 = (27)^2 = 729$

(d) $\lim_{x\to c}[f(x)]^{\frac{2}{3}} = \left(\lim_{x\to c} f(x)\right)^{2/3} = (27)^{2/3} = 9$

THEOREM 1.6 Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$

Ex.5 Evaluate: $\lim_{x \to \frac{5\pi}{3}} \cos(x)$

$= \cos\left(\frac{5}{3}\pi\right)$

$= \frac{1}{2}$

THEOREM 1.7 Functions That Agree at All But One Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

Ex.6 Evaluate: $\lim_{x \to -2} \frac{3x^2 + 5x - 2}{x + 2}$

$= \lim_{x \to -2} \frac{(x+2)(3x-1)}{x+2} = \lim_{x \to -2} (3x-1) = -7$

$= \lim_{x \to -2} 3x - \lim_{x \to -2} 1$

$= 3 \lim_{x \to -2} x - 1$

$= 3 \cdot (-2) - 1$

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. If the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.]
3. Apply Theorem 1.7 to conclude *analytically* that

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c).$$

4. Use a *graph* or *table* to reinforce your conclusion.

Ex.7 Evaluate: $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$

conjugates?

$$= \lim_{x\to 3}\left[\frac{\sqrt{x+1}-2}{x-3}\right]\cdot\left[\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right]$$

$$= \lim_{x\to 3} \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{(x-3)\cdot 1}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2}$$

$$= \frac{1}{\sqrt{(3)+1}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$= \frac{1}{2+2}$$

$$= \frac{1}{4} \checkmark$$

Ex.8 Evaluate: $\lim\limits_{x\to 0} \dfrac{\frac{1}{x+4} - \frac{1}{4}}{x}$ $= \lim\limits_{x\to 0} \dfrac{\frac{1}{x+4} - \frac{1}{4}}{\frac{x}{1}} \cdot \dfrac{\frac{(x+4)\cdot 4}{1}}{\frac{(x+4)\cdot 4}{1}}$

$$= \lim_{x\to 0} \frac{\frac{4\cancel{(x+4)}}{\cancel{x+4}} - \frac{1\cdot\cancel{4}(x+4)}{\cancel{4}}}{\frac{x}{1}\cdot\frac{4\cdot(x+4)}{1}}$$

$$= \lim_{x\to 0} \frac{4 - x - 4}{x\cdot 4\cdot(x+4)}$$

$$= \lim_{x\to 0} \frac{-\cancel{x}}{\cancel{x}\cdot 4\cdot(x+4)}$$

$$= \lim_{x\to 0} \frac{-1}{4\cdot(x+4)}$$

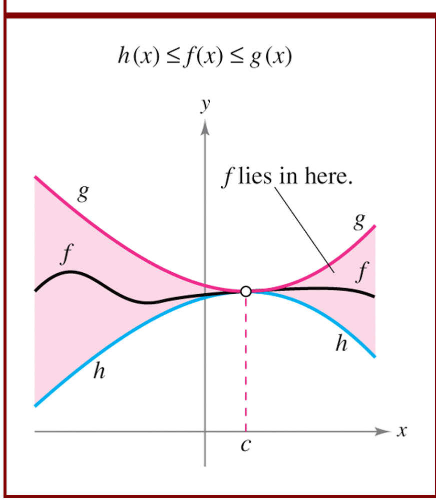
$$= \frac{-1}{4(0+4)} = -\frac{1}{16} \quad ✓$$

THEOREM 1.8 The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim\limits_{x\to c} f(x)$ exists and is equal to L.



Ex.9 Evaluate: $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right)$

We know $-1 \le \cos\left(\frac{1}{x^2}\right) \le +1$ for all $x \in (-\infty, \infty)$

$-1 \cdot x^2 \le x^2 \cdot \cos\left(\frac{1}{x^2}\right) \le +1 \cdot x^2$, since $x^2 \ge 0$

$-x^2 \le x^2 \cos\left(\frac{1}{x^2}\right) \le x^2$

$\lim_{x\to 0}(-x^2) \le \lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) \le \lim_{x\to 0}(x^2)$

$0 \le \lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) \le 0$

So, $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$

THEOREM 1.9 Two Special Trigonometric Limits

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ 2. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Ex.10 Evaluate: $\lim_{x\to 0} \frac{\cos(x) - \sin(x) - 1}{2x} = -\frac{1}{2}$ ✓

$= \lim_{x\to 0} \frac{\cos(x) - 1}{2x} - \lim_{x\to 0} \frac{\sin(x)}{2x}$.

$= -\frac{1}{2} \cdot \lim_{x\to 0} \left(\frac{1-\cos(x)}{x}\right) - \frac{1}{2} \cdot \lim_{x\to 0} \frac{\sin(x)}{x}$

$= -\frac{1}{2} \cdot 0 \quad -\frac{1}{2} \cdot 1$

$= 0 - \frac{1}{2}$

$= -\frac{1}{2}$ ✓

$f(\square) = 3\square^2 + 1\square$ $\qquad$ Δx one symbol

Ex.11 Given $f(x) = 3x^2 + x$, evaluate: $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[3(x+\Delta x)^2 + (x+\Delta x)] - [3x^2 + x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x^2 + 2x\Delta x + (\Delta x)^2) + x + \Delta x - 3x^2 - x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 + 6x\Delta x + 3(\Delta x)^2 + \Delta x - 3x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3(\Delta x)^2 + \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\cancel{\Delta x}}{\cancel{\Delta x}} + \lim_{\Delta x \to 0} \frac{3(\Delta x)^{\cancel{2}}}{\cancel{\Delta x}} + \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} (6x) + \lim_{\Delta x \to 0} 3\Delta x + \lim_{\Delta x \to 0} 1$$

$$= 6x + 0 + 1$$

$$= 6x + 1 \checkmark$$

FOIL

$(x+\Delta x)^2$

$= (x+\Delta x)(x+\Delta x)$

$= x^2 + x\cdot\Delta x + x\cdot\Delta x + (\Delta x)^2$

$= x^2 + 2x\Delta x + (\Delta x)^2$